Notes compiled by Attila Egri-Nagy for the Category Theory Reading Club using the book *The Joy of Abstraction* by Eugenia Cheng. (Cheng, 2022)

Contents

Abstract Ideas – From the Familiar to the Unknown

Using abstract ideas is a skill shared by everyone. It is not only a privilege of mathematicians. Mathematical intuition develops with (the right kind of) practice.

Abstraction

The act of *forgetting details* that are irrelevant in a given context. It enables understanding and problem solving by reducing the cognitive load.

Hence, the "the joy of abstraction" is the joy of understanding.

Here, we mention three examples of abstract concepts: numbers, sets, and categories. They are in decreasing order of familiarity. Everyone knows numbers, everyone is taught about sets in school, and only a very few heard about categories. The point is that numbers of already very abstract, so there is no need to be afraid of categories!

Counting: Natural Numbers

Everyone is familiar with the counting numbers.

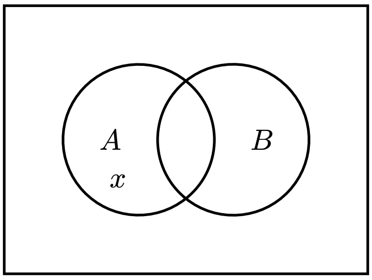
$$0, 1, 2, 3, \dots$$

We learn those as little kids, thus we do not realize the magnitude of the achievement. The idea of number two is already very abstract. It is not two apples, or two dogs. These are just examples. The symbol 2 is just a convenient shorthand. The point, the very idea of two-ness, is having a thing and another one in a collection.

It is remarkable that how effortlessly we can work with these numbers. We do not need to carry visual aids with us, like three apples, or at least an image of them to recognize other collections with three objects. Number 3 is just here in our mind, immediately. It is *intuitive*.

Belonging to a Collection: Set Membership

Another abstraction that we learn in school is *set membership*, the idea that a thing belongs to a collection. We say x is a member of the set A, denote this by $x \in A$, and visualize the situation by diagrams.



Set theory has been the officially chosen foundations for mathematics. Mathematical objects are defined as sets with some properties. It is somewhat forced. Even functions, that feel more like processes, are defined as subsets of the direct product of two sets. It is a very static description. Nevertheless, the concept still based on a very immediate and natural experience: things in a collection, like apples in a bag.

Composable Relationships: Category

Set membership is a special case of relations. In general, we can say that x is related to y.

$$x \longrightarrow y$$

This relation can be many things,

- x loves y,
- there is a road from x to y,
- x and y has a similar shape (x can be morphed into y),
- $x \leq y$,
- …,

just to name a few. The variety of relations is bewildering. We do not even call them relations, just *arrows* between *objects*. It is a very abstract concept, so it fits many situations. Please note, that numbers are already very abstract, they fit any situations with distinguishable objects.

Can we have just any relations? No, in order to have a category, they should composable. If we can match two arrows head to tail,

$$x \longrightarrow y \longrightarrow z$$

then there has to be a direct arrow between from x to z.

$$\begin{tikzcd} x \arrow[r] \arrow[rr, bend left] & y \arrow[r] & z \end{tikzcd}$$

In other words, *composition* should work for a collection of objects and arrows in order to call them a *category*.[1]

It turns out that these collections of objects and arrows that behave well can describe most of mathematics (as an organizing principle). More interestingly, their applicability goes well beyond mathematics itself. That's why it is worth getting familiar with them.

[1] There is a bit more technicality in the definition, but this is the core idea.

... in mathematics you don't understand things. You just get used to them.[2]

— John von Neumann

Role in a context

Categorical Abstraction

In category theory, we forget the internal structure and properties of objects. We focus on what relationships they have with other objects, i.e., their role in the given context.

We look at two different contexts that have the same 'shape', i.e., the network of relationships. Then, we will find that in both contexts there are objects playing the same role, though the objects are different.

The context of multiplication

In the context of multiplication the *prime* numbers are the basic building blocks. We create *composite* numbers by multiplying primes together. Figure 1 shows how multiplying 1 (the 'origin') by 3 prime numbers at most once gives a cube-like shape. Multiplication by a particular number corresponds to going a unit distance in that particular direction. We have 3 numbers, so we have 3 dimensions.

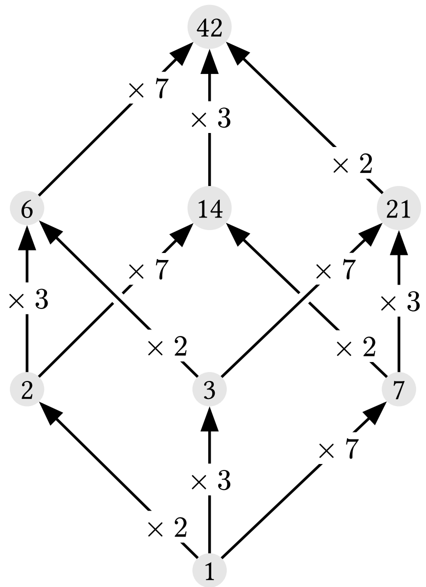


Figure 1: The context of multiplying 1 by 2,3, or 7, at most once. An arrow means 'being a divisor of'. Not all arrows are shown!

The context of adding to a collection

Adding new items to a collection also has a cube-like shape if we have 3 items. Figure 2 shows that with three fruits. The empty set $\{\}$ is often denoted by $\varnothing$.

[2] This is quite a popular quote, although no one knows what he meant exactly (see https://math.stackexchange.com/questions/11267/what-are-some-interpretations-of-von-neumanns-quote). I think the relevant reading here is something like: don't worry about abstract, scary looking mathematical concepts. You spend some time with them and they will be familiar, friendly ideas.

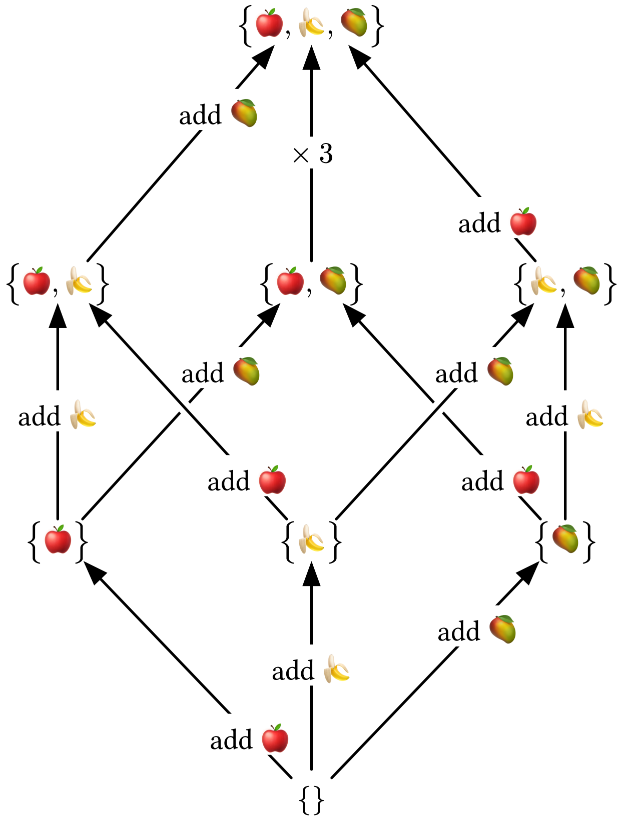


Figure 2: The context of collecting types of fruits. An arrow means 'being a subset of'.

Same role in different contexts

Now we say that the number 1 plays the same role in the multiplication context as the empty set $\varnothing$ in the collection extending context. *What is common in 1 and $\varnothing$?* Not much. 1 is a number and $\varnothing$ is an empty container. They are not of the same type, and they do not share properties. But, they are both at the bottom in their contexts. Indeed, they share the relational property of being related to everything else in the context.[3] 1 can divide any number. Similarly, $\varnothing$ is the subset of all other sets.

An Analogy: The comedian in the class

In school you might have had a classmate who always made everyone laugh, liked by everyone. Now, you are at university and one course has a similar resident comedian. Obviously, the two persons are totally different. Name, gender, hair and eye color do not transfer between the two situations. But the role is the same, making everyone laugh. Category theory is interested only in the abstract role, not in the individual properties.[4] This is the core of categorical thinking.

Same object, different roles in different contexts

Take number 1. Regarding addition, it has the property that it *generates* all natural numbers: starting from zero, by keep adding 1 we can reach all numbers.

Regarding multiplication, it plays a totally different role. Multiplying by 1 has no effect. IF we want to generate all numbers, we need all the prime numbers. This shows that numbers with addition and numbers with multiplication are two genuinely different contexts.

[3] Such an object is called *initial*.

[4] Ultimately, the theory also says that looking at all the relationships is exactly the same as examining all the intrinsic properties (Yoneda Lemma).

On mathematical notation & terminology

The issue of mathematical notation is seldom mentioned, though

- it is crucial, externalized (written down) mathematics is just notation,
- it can lead to misunderstandings.

The preface of (Barr & Wells, 1995) is honest about it:

> *"In most scientific disciplines, notation and terminology are standardized, often by an international nomenclature committee. (Would you recognize Einstein's equation if it said $p = HU^2$?) We must warn the nonmathematician reader that such is not the case in mathematics. There is no standardization body and terminology and notation are individual and often idiosyncratic. We will introduce and stick to a fixed notation in this book, but any reader who looks in another source must expect to find different notation and even different names for the same concept – or what is worse, the same name used for a different concept."*

There are negative and positive readings of this. Negatively, the lack of standardized notation is a source of confusion and makes the learning process more difficult. Positively, we can interpret it in a liberating way: as long as we pay attention to the communication process (e.g., being clearly defined and consistent), we are free to choose creatively our notation.

Understanding abstract definitions

Possibly the easiest method to understand the definition of a category is to pick a familiar example. The most accessible is probably the category of sets, where the objects are sets and the arrows represent functions between those sets. With a concrete representation, we can look into an object or into an arrow. On the abstract level, we just have objects and arrows.

By looking at just one example, it may be unclear what the category theory formulation provides. One has to look at other examples.

> *"It is typical of most of the useful concepts in mathematics that there is more than one way of perceiving or understanding them. It is simply not true that everything about a mathematical concept is contained in its definition. Of course it is true that in some sense all the theorems are inherent in its definition, but not what makes it useful to mathematicians or to scientists who use mathematics. We believe that the more ways you have of perceiving an idea, the more likely you are to recognize situations in your own work where the idea is useful."*
>
> — (Barr & Wells, 1995)

It is also possible to understand a definition abstractly. We need to remember the things and their properties exactly as we cannot simply think of our favourite concrete example to recall those properties. This requires a bit more discipline.

Bibliography

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